Dynamic moment invariants for nonlinear Hamiltonian systems

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Distributions of particles being transported through a nonlinear Hamiltonian system are studied. Using normal form techniques, a procedure to obtain invariant functions of moments of the distribution is given. These functions are invariant for the given Hamiltonian system and are called dynamic moment invariants. These techniques are used to obtain dynamic moment invariants for the nonlinear pendulum Hamiltonian system. [S1063-651X(99)13703-6]

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I. INTRODUCTION

We consider a particle distribution being transported under the action of a nonlinear Hamiltonian system. It would be useful to obtain quantities that remain invariant under this transport. To this end, we study invariant functions of moments of the distribution. Functions of moments that remain invariant for all linear Hamiltonian systems (belonging to the class of kinematic invariants) have already been constructed [1–3]. Dynamic moment invariants (which remain invariant for a given Hamiltonian system) have been investigated for linear systems [3]. In this paper, we construct dynamic moment invariants for nonlinear Hamiltonian systems.

In Sec. II, we provide a brief background to Lie algebraic methods and moments of distribution. In Sec. III, we outline a procedure for constructing dynamic moment invariants. This makes use of normal form techniques for symplectic maps. In Sec. IV, we construct dynamic moment invariants for the nonlinear pendulum Hamiltonian. We first obtain a symplectic map corresponding to this system, and compute its normal form. Then we apply techniques described in Sec. III to obtain the dynamic moment invariants. Our conclusions are given in Sec. IV.

II. PRELIMINARIES

In this section we briefly describe the Lie algebraic tools [4] and concepts required for our purpose. We also introduce moments of a particle distribution and describe their transport under the action of a general Hamiltoninan system.

We start by defining Lie operators. Let us denote the collection of $2n$ phase-space variables q_i, p_i $(i=1,2,3,\ldots,n)$ by the symbol z:

$$z=(q_1,p_1,q_2,p_2\ldots,q_n,p_n). \tag{2.1}$$

The Lie operator corresponding to a phase-space function $f(z)$ is denoted by $:f(z):$. It is defined by its action on a phase-space function $g(z)$ as shown below:

$$:f(z):g(z)=[f(z),g(z)], \tag{2.2}$$

where $[f(z),g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$. Let f_r be a homogeneous polynomial in z of degree r. Then $[f_r,f_s]$ is a homogeneous polynomial of degree $(r+s-2)$. Next we define the exponential of a Lie operator. It is called a Lie transformation, and is given as follows:

$$e^{:f(z):}=\sum_{n=0}^{\infty}\frac{:f(z):^n}{n!}. \tag{2.3}$$

Consider the action of a (nonlinear) Hamiltonian system specified by the Hamiltonian $H(z,t)$ on a particle. Let z^i denote the phase-space coordinates of the particle at time t^i, and z^f the coordinates at some final time t^f. Then one can find a map $\mathcal{M}$, such that

$$z^f=\mathcal{M}z^i. \tag{2.4}$$

It can be shown [4] that this map is symplectic (i.e., its Jacobian matrix M satisfies the symplectic condition $\tilde{M}JM=J$, where J is the fundamental symplectic matrix). By the Dragt-Finn factorization theorem, $\mathcal{M}$ can be written as an infinite product of Lie transformation in the form [4]

$$\mathcal{M}=\hat{M}e^{:f_3:}e^{:f_4:}\cdots \tag{2.5}$$

where $\hat{M}$ is the Lie transformation corresponding to the Jacobian matrix M [i.e., $(\hat{M}z)_i=M_{ij}z_j$]. Each function f_m is a homogeneous polynomial of degree m.

We have considered the action of a Hamiltonian system on a single particle. We now consider its action on a particle distribution. Let $h(z)$ be a distribution function describing particle density in phase space at the point having coordinates z. We assume that the particles do not interact with one another (alternatively, the effects due to particle interactions should be treatable by the Vlasov approximation). Again, representing the Hamiltonian system in terms of a symplectic map $\mathcal{M}$, we can show that [4,3]

$$h^{\rm fin}(z)=h^{\rm in}(\mathcal{M}^{-1}z), \tag{2.6}$$

where $h^{\rm in}$ and $h^{\rm fin}$ are the initial and final distributions, respectively.

Next we define moments of the above distribution, and determine their evolution under the action of the Hamil-

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tonian system (or equivalently the corresponding symplectic map $\mathcal{M}$). Let $\{P_\alpha(z)\}[\alpha=1,2,3,\ldots,N(m)]$ denote the set of basis monomials of a fixed degree m in the $2n$ phase-space variables where $N(m)={}^{m+2n-1}C_m$. Each $P_\alpha(z)$ corresponds to a unique mth-order basis moment $\langle P_\alpha(z)\rangle$ through the relation:

$$\langle P_\alpha(z)\rangle=\int d^{2n}z'h(z')P_\alpha(z'). \tag{2.7}$$

The initial and final moments of the distribution before and after transport through a Hamiltonian system are therefore given by

$$\langle P_\alpha(z)\rangle^{\text{in}}=\int d^{2n}z'h^{\text{in}}(z')P_\alpha(z'), \tag{2.8}$$

$$\langle P_\alpha(z)\rangle^{\text{fin}}=\int d^{2n}z'h^{\text{fin}}(z')P_\alpha(z'). \tag{2.9}$$

Using the relation between the initial and final distribution given in Eq. (2.6), we can give a relation between the initial and final moments [3]:

$$\langle P_\alpha(z)\rangle^{\text{fin}}=\int d^{2n}z'h^{\text{in}}(z')P_\alpha(\mathcal{M}z'). \tag{2.10}$$

We write this formally as

$$\langle P_\alpha(z)\rangle^{\text{fin}}=\mathcal{M}\langle P_\alpha(z)\rangle^{\text{in}}. \tag{2.11}$$

Since $\mathcal{M}$ can be expressed in terms of Lie operators, it is useful later on to have the following relation (see Appendix C of Ref. [3])

$$:f:\langle P_\alpha(z)\rangle=\langle :f:P_\alpha(z)\rangle. \tag{2.12}$$

III. DYNAMIC MOMENT INVARIANTS FOR HAMILTONIAN SYSTEMS

A. General concepts

A function of moments that remains invariant under transport through all Hamiltonian systems is called a kinematic moment invariant. Kinematic moment invariants for linear Hamiltonian systems have already been obtained [1,3]. We call functions of moments that remain invariant only under the action of a particular Hamiltonian system (or the equivalent symplectic map) dynamic moment invariants. Dynamic moment invariants for linear Hamiltonians have already been derived [3]. In this paper, we consider dynamic moment invariants for nonlinear Hamitonians.

Consider a nonlinear Hamiltonian system described by the symplectic map $\mathcal{M}$. We denote a dynamic moment invariant that is a function of kth-order moments by $\mathcal{D}_k$. Since $\mathcal{D}_k$ is invariant under the action of $\mathcal{M}$,

$$\mathcal{D}_k=\mathcal{M}\mathcal{D}_k. \tag{3.1}$$

To construct these dynamic moment invariants, we consider the normal form [5,4] of $\mathcal{M}$:

$$\mathcal{M}=\mathcal{A}^{-1}\mathcal{N}\mathcal{A}, \tag{3.2}$$

where

$$\mathcal{N}=e^{:\eta:}. \tag{3.3}$$

For a nonresonant Hamiltonian [6],

$$\eta=f(J_1,J_2,J_3,\ldots,J_n), \tag{3.4}$$

where

$$J_i=\frac{(q_i^2+p_i^2)}{2}. \tag{3.5}$$

First, we find a function $\mathcal{I}_k$ that is invariant under $\mathcal{N}$. That is,

$$\mathcal{I}_k=\mathcal{N}\mathcal{I}_k, \tag{3.6}$$

where $\mathcal{I}_k$ is a function of kth order moment invariants. Let

$$\mathcal{D}_k\equiv\mathcal{A}^{-1}\mathcal{I}_k. \tag{3.7}$$

Then, using Eqs. (3.2), (3.6), and (3.7), we obtain

$$\begin{aligned}\mathcal{M}\mathcal{D}_k&=\mathcal{A}^{-1}\mathcal{N}\mathcal{A}\mathcal{D}_k\\&=\mathcal{A}^{-1}\mathcal{N}\mathcal{A}\mathcal{A}^{-1}\mathcal{I}_k\\&=\mathcal{A}^{-1}\mathcal{N}\mathcal{I}_k=\mathcal{A}^{-1}\mathcal{I}_k=\mathcal{D}_k.\end{aligned} \tag{3.8}$$

Thus we see that $\mathcal{D}_k$ is a dynamic moment invariant of the map $\mathcal{M}$. Note that the above procedure has been used previously to find functions of phase-space variables that are invariant under the action of a symplectic map [4]. Here we have extended this to moments.

B. Normal form moment invariants

From the above discussion it is obvious that the first step toward finding dynamic invariants is to find moment invariants $\mathcal{I}_k$ for symplectic maps in the normal form

$$e^{:\eta:}\mathcal{I}_k=\mathcal{I}_k, \tag{3.9}$$

which implies

$$\left(1+:\eta:+\frac{:\eta:^2}{2}+\cdots\right)\mathcal{I}_k=\mathcal{I}_k. \tag{3.10}$$

Thus a sufficient condition for the above equation to be true is

$$:\eta:\mathcal{I}_k=0. \tag{3.11}$$

For a nonresonant Hamiltonian, we have already seen that η is a function only of the J_i's. In this case, the above equation is satisfied if the following annihilation conditions hold:

$$:J_i:\mathcal{I}_k=0,\quad i=1,2,3,\ldots,n. \tag{3.12}$$

In fact, $\mathcal{I}_k$ is a function of moments that is invariant under $U(1)\otimes U(1)\otimes\cdots\otimes U(1)$, taken n times. For convenience, we will call $\mathcal{I}_k$ a ‘‘kth-order normal form moment invariant.’’

With this brief background, we are in a position to construct the normal form moment invariants. Consider the set of basis monomials of a fixed degree m, $\{P_\alpha(z)\}$, $\{\alpha=1,2,\ldots,N(m)\}$. A general element of $\{P_\alpha(z)\}$ is given by the relation

$$P_\alpha(z)=\prod_{i=1}^{n} q_i^{r^{(i)}} p_i^{s^{(i)}}, \tag{3.13}$$

where $r^{(1)}+s^{(1)}+\cdots+r^{(n)}+s^{(n)}=m$. There is a unique mth-order basis moment associated with each $\{P_\alpha(z)\}$ through the relation given in Eq. (2.7). Let $\mathcal{I}_m^{(l)}$ be a function of mth-order moments defined as follows:

$$\mathcal{I}_m^{(l)}=A_j W_j^{(m)}, \tag{3.14}$$

where the repeated index j is summed over. The coefficients A_j's are constants and $W_j^{(m)}$'s are the product of l mth-order basis moments. Therefore, $W_j^{(m)}$'s are of the form

$$\langle P_{\alpha_1}(z)\rangle\langle P_{\alpha_2}(z)\rangle\cdots\langle P_{\alpha_l}(z)\rangle, \tag{3.15}$$

where $\alpha_1\leqslant\alpha_2\leqslant\cdots\leqslant\alpha_l$.

For $\mathcal{I}_m^{(l)}$ to be a normal form moment invariant, it has to satisfy the following equation [cf. Eq. (3.12)]:

$$:J_i:\mathcal{I}_m^{(l)}=A_j:J_i:W_j^{(m)}=0, \quad i=1,2,\ldots,n. \tag{3.16}$$

Thus our task reduces to determining the constants A_j's such that the above equation is satisfied. A moment invariant of the form $\mathcal{I}_m^{(l)}$ is called a pure invariant, since it is a function of moments of the same order m. One can also derive mixed invariants which are functions of moments of different orders.

C. Examples

In this subsection, we derive explicit expressions for various normal form moment invariants to illustrate how these invariants are found. First, we derive the expression for $\mathcal{I}_2^{(1)}$. We restrict ourselves to two-dimensional phase space for simplicity. The most general form of $\mathcal{I}_2^{(1)}$ is then given by

$$\mathcal{I}_2^{(1)}=A_1\langle q_1^2\rangle+A_2\langle p_1^2\rangle+A_3\langle q_1p_1\rangle, \tag{3.17}$$

where A_1, A_2, and A_3 are as yet undetermined constants. These constants are fixed by ensuring that the following annihilation condition is satisfied [cf. Eq. (3.16)]:

$$:J_1:\mathcal{I}_2^{(1)}=0, \tag{3.18}$$

where

$$J_1=\frac{q_1^2+p_1^2}{2}. \tag{3.19}$$

That is,

$$:q_1^2:\mathcal{I}_2^{(1)}+:p_1^2:\mathcal{I}_2^{(1)}=0. \tag{3.20}$$

The above condition implies that $A_2=A_1$ (where A_1 is arbitrary) and $A_3=0$. Thus we finally obtain

$$\mathcal{I}_2^{(1)}=A_1(\langle q_1^2\rangle+\langle p_1^2\rangle). \tag{3.21}$$

The expressions for $\mathcal{I}_2^{(1)}$ in four- and six-dimensional phase space, respectively, are given as follows (here the constants A_1, A_2, and A_3 are arbitrary):

$$A_1(\langle q_1^2\rangle+\langle p_1^2\rangle)+A_2(\langle q_2^2\rangle+\langle p_2^2\rangle), \tag{3.22}$$

$$A_1(\langle q_1^2\rangle+\langle p_1^2\rangle)+A_2(\langle q_2^2\rangle+\langle p_2^2\rangle)+A_3(\langle q_3^2\rangle+\langle p_3^2\rangle). \tag{3.23}$$

Using the above procedure, we can easily derive expressions for other quadratic moment invariants $\mathcal{I}_2^{(l)}$, for higher values of l. For a two-dimensional phase space, two of them are given below (here the constants A_1 and A_2 are arbitrary):

$$\mathcal{I}_2^{(2)}=A_1(\langle q_1^2\rangle^2+\langle p_1^2\rangle^2)+A_2\langle q_1^2\rangle\langle p_1^2\rangle+(2A_1-A_2)\langle q_1p_1\rangle^2, \tag{3.24}$$

$$\mathcal{I}_2^{(3)}=A_1(\langle q_1^2\rangle^3+\langle p_1^2\rangle^3)+A_2(\langle q_1^2\rangle\langle p_1^2\rangle^2+\langle q_1^2\rangle^2\langle p_1^2\rangle)+(3A_1-A_2)(\langle q_1^2\rangle+\langle p_1^2\rangle)\langle q_1p_1\rangle^2. \tag{3.25}$$

Next we derive the expression for $\mathcal{I}_3^{(2)}$ where each term is a product of two cubic moments. The most general form of $\mathcal{I}_3^{(2)}$ (in a two-dimensional phase space) is given by

$$\begin{aligned}\mathcal{I}_3^{(2)}=&A_1\langle p_1^3\rangle^2+A_2\langle p_1^3\rangle\langle q_1p_1^2\rangle+A_3\langle p_1^3\rangle\langle q_1^2p_1\rangle+A_4\langle q_1^3\rangle\langle p_1^3\rangle+A_5\langle q_1p_1^2\rangle^2+A_6\langle q_1p_1^2\rangle\langle q_1^2p_1\rangle\\&+A_7\langle q_1p_1^2\rangle\langle q_1^3\rangle+A_8\langle q_1^2p_1\rangle^2+A_9\langle q_1^2p_1\rangle\langle q_1^3\rangle+A_{10}\langle q_1^3\rangle^2.\end{aligned} \tag{3.26}$$

Applying Eq. (3.16), we obtain

$$\begin{aligned}&(6A_1-2A_3-2A_5)\langle p_1^3\rangle\langle q_1p_1^2\rangle+(2A_2-3A_4-A_6)\langle p_1^3\rangle\langle q_1^2p_1\rangle-A_2\langle p_1^3\rangle^2+(3A_2-2A_6)\langle q_1p_1^2\rangle^2+(A_3-A_7)\langle p_1^3\rangle\langle q_1^3\rangle\\&\quad+(3A_3+4A_5-3A_7-4A_8)\langle q_1^2p_1\rangle\langle q_1p_1^2\rangle+(3A_4+A_6-2A_9)\langle q_1^3\rangle\langle q_1p_1^2\rangle+(2A_6-3A_9)\langle q_1^2p_1\rangle^2\\&\qquad+(2A_7+2A_8-6A_{10})\langle q_1^3\rangle\langle q_1^2p_1\rangle+A_9\langle q_1^3\rangle^2=0.\end{aligned} \tag{3.27}$$

Setting the coefficient of each independent term to be zero, we obtain

$$A_2=0, \quad A_1=A_{10}, \quad A_3=A_7,$$
$$A_4=0, \quad A_5=3A_1-A_3, \tag{3.28}$$
$$A_6=0, \quad A_8=A_5, \quad A_9=0.$$

Thus we finally obtain

$$\mathcal{I}_3^{(2)}=A_1(\langle q_1^3\rangle^2+\langle p_1^3\rangle^2)+(3A_1-A_3)(\langle q_1p_1^2\rangle^2+\langle q_1^2p_1\rangle^2)A_3(\langle q_1^3\rangle\langle q_1p_1^2\rangle+\langle p_1^3\rangle\langle q_1^2p_1\rangle), \tag{3.29}$$

where A_1 and A_3 are arbitrary constants.

Using the above procedure, we can also derive the following quartic moment invariants (again, A_1 and A_2 are arbitrary constants):

$$\mathcal{I}_4^{(1)}=A_1(\langle q_1^4\rangle+\langle p_1^4\rangle+2\langle q_1^2p_1^2\rangle), \tag{3.30}$$

$$\mathcal{I}_4^{(2)}=A_1(\langle q_1^4\rangle^2+\langle p_1^4\rangle^2)+A_2(\langle q_1^4\rangle+\langle p_1^4\rangle)\langle q_1^2p_1^2\rangle+\tfrac{1}{2}A_2\langle q_1^4\rangle\langle p_1^4\rangle$$
$$+\tfrac{1}{2}(12A_1-A_2)\langle q_1^2p_1^2\rangle^2+(4A_1-A_2)(\langle q_1^3p_1\rangle^2+\langle q_1p_1^3\rangle^2). \tag{3.31}$$

D. Functionally independent invariants

In this subsection, we enumerate the functionally independent normal form moment invariants. We observe that there is only one functionally independent invariant in the moment invariants of the form $\mathcal{I}_2^{(1)}$, viz.

$$\langle q_1^2\rangle+\langle p_1^2\rangle. \tag{3.32}$$

All the other $\mathcal{I}_2^{(1)}$ invariants are obtained by multiplying the above equation by a scalar. For the $\mathcal{I}_2^{(2)}$ invariants, there are only two functionally independent invariants, viz.

$$\langle q_1^2\rangle^2+\langle p_1^2\rangle^2+2\langle q_1p_1\rangle^2 \tag{3.33}$$

and

$$\langle q_1^2\rangle\langle p_1^2\rangle-\langle q_1p_1\rangle^2. \tag{3.34}$$

These are obtained by assigning $A_1=1$, $A_2=0$ and $A_1=0$, $A_2=1$, respectively, in Eq. (3.24). All the other $\mathcal{I}_2^{(2)}$ invariants are obtained by a linear combination of the above two functionally independent moment invariants.

Using the above procedure, we find that there are two functionally independent $\mathcal{I}_3^{(2)}$ invariants given by

$$\langle q_1^3\rangle^2+\langle p_1^3\rangle^2+3\langle q_1^3\rangle\langle q_1p_1^2\rangle+3\langle p_1^3\rangle\langle q_1^2p_1\rangle \tag{3.35}$$

and

$$\langle q_1p_1^2\rangle^2+\langle q_1^2p_1\rangle^2-\langle q_1^3\rangle\langle q_1p_1^2\rangle-\langle p_1^3\rangle\langle q_1^2p_1\rangle. \tag{3.36}$$

The single functionally independent $\mathcal{I}_4^{(1)}$ invariant is given by

$$\langle q_1^4\rangle+\langle p_1^4\rangle+2\langle q_1^2p_1^2\rangle, \tag{3.37}$$

and the two functionally independent $\mathcal{I}_4^{(2)}$ invariants are given by

$$\langle q_1^4\rangle^2+\langle p_1^4\rangle^2+6\langle q_1^2p_1^2\rangle^2+4\langle q_1^3p_1\rangle^2+4\langle q_1p_1^3\rangle^2 \tag{3.38}$$

and

$$\langle q_1^4\rangle+\langle p_1^4\rangle\langle q_1^2p_1^2\rangle+\tfrac{1}{2}\langle q_1^4\rangle\langle p_1^4\rangle$$
$$-\tfrac{1}{2}\langle q_1^2p_1^2\rangle^2-\langle q_1^3p_1\rangle^2-\langle q_1p_1^3\rangle^2. \tag{3.39}$$

Similarly, we can easily find the functionally independent moment invariants of the other $\mathcal{I}_m^{(l)}$'s. Once we have the above normal form invariants, we can easily find the corresponding dynamic invariants for a particular Hamiltonian system by using Eq. (3.7).

IV. DYNAMIC MOMENT INVARIANTS FOR THE NONLINEAR PENDULUM HAMILTONIAN

In this section, we will obtain the dynamic moment invariants for a distribution of particles going through a potential given by the pendulum Hamiltonian. We consider the pendulum Hamiltonian, since it is a prototypical nonlinear Hamiltonian system. Further, explicit results can be written down in this case. But the method outlined in this paper is equally applicable to any other nonlinear Hamiltonian system. The expressions for moment invariants, however, are complicated for a general system, and hence cannot be written in a compact form. For this reason, we have not considered such systems in this paper even though the full power of the Lie algebraic approach becomes evident only in such cases. For such complicated systems, no analytic solutions are available and the invariants cannot be written down by direct methods. Thus a perturbative (Lie algebraic) approach outlined below in this section is the only viable solution. To obtain the dynamic moment invariants, we first need the symplectic map corresponding to the pendulum Hamiltonian.

A. Pendulum map

We consider the pendulum Hamiltonian $H(q_1,p_1)$,

$$H(q_1,p_1)=\frac{1}{2}p_1^2-\cos(q_1)+1. \tag{4.1}$$

On expanding, we have

$$H(q_1,p_1)=H_2+H_4+\cdots, \tag{4.2}$$

where

$$H_2=\frac{1}{2}p_1^2+\frac{1}{2}q_1^2, \quad H_4=-\frac{1}{4!}q_1^4. \tag{4.3}$$

The symplectic map $\mathcal{M}$ corresponding to the pendulum Hamiltonian is given by

$$\mathcal{M}=\hat{M}e^{:f_3:}e^{:f_4:}\cdots. \tag{4.4}$$

The Jacobian matrix M is given by a 2×2 rotation matrix [7]:

$$M=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{4.5}$$

Since $H_{2n+1}=0$ $(n=1,2,\ldots)$, we have $f_{2n+1}=0\,(n=1,2,\ldots)$. In particular, $f_3=0$. The fourth degree polynomial f_4 is given by [4,7]

$$f_4=\int_0^t dt' H_4(M^{-1}z)=\frac{1}{24}\int_0^t dt'(q_1\cos t'-p_1\sin t')^4. \tag{4.6}$$

On integrating we obtain

$$f_4=c_1q_1^4+c_2p_1^4+c_3q_1^3p_1+c_4q_1^2p_2+c_5q_1p_1^3, \tag{4.7}$$

where

$$c_1=\tfrac{1}{192}(3t+\tfrac{1}{4}\sin 4t+2\sin 2t),$$

$$c_2=\tfrac{1}{192}(3t+\tfrac{1}{4}\sin 4t-2\sin 2t), \tag{4.8}$$

$$c_3=-\tfrac{1}{24}(1-\cos^4 t), \quad c_4=\tfrac{1}{128}(4t-\sin 4t),$$

$$c_5=-\tfrac{1}{24}(\sin^4 t). \tag{4.9}$$

In principle, we can also calculate the higher order f_m's. For simplicity, we restrict ourselves to fourth order.

B. Normal Form of the Pendulum map

We now calculate the normal form $\mathcal{N}=\mathcal{A}MA^{-1}$ of the pendulum map (up to fourth order) and its corresponding transformation $\mathcal{A}$. This is the next step in calculating the dynamic moment invariants for the pendulum map. Since the linear part $\hat{M}$ is already in the normal form, we consider a transformation of the type

$$\mathcal{A}=e^{:h_4:}, \tag{4.10}$$

where h_4 is a homogeneous polynomial of fourth degree.

To simplify the determination of a_i's, we introduce the real resonance basis [4]. We define the following:

$$|l,k\rangle_R=\frac{1}{2}((q_1+ip_1)^l(q_1-ip_1)^k+(q_1-ip_1)^l(q_1+ip_1)^k), \tag{4.11}$$

$$|l,k\rangle_I=\frac{i}{2}(-(q_1+ip_1)^l(q_1-ip_1)^k + (q_1-ip_1)^l(q_1+ip_1)^k), \tag{4.12}$$

where $l+k=n$ and $l\geqslant k$. Using the above real resonance basis, Eq. (4.7) can be written as follows:

$$f_4=c_{40}|4,0\rangle_R+c_{31}|3,1\rangle_R+c_{22}|2,2\rangle_R + d_{40}|4,0\rangle_I+d_{31}|3,1\rangle_I, \tag{4.13}$$

where

$$c_{40}=\frac{1}{8}(c_1+c_2-c_4)=\frac{\sin 4t}{768},$$

$$c_{31}=\frac{1}{2}(c_1-c_2)=\frac{\sin 2t}{96}, \tag{4.14}$$

$$c_{22}=\frac{1}{8}(3c_1+3c_2+c_4)=\frac{t}{64},$$

$$d_{40}=\frac{1}{8}(c_3-c_5)=\frac{-(\sin 2t)^2}{384}, \tag{4.15}$$

$$d_{31}=\frac{1}{4}(c_3+c_5)=\frac{\cos 2t-1}{96}. \tag{4.16}$$

Now the normal form of the pendulum map can be written as follows:

$$\begin{aligned}\mathcal{N}&=\mathcal{A}\mathcal{M}\mathcal{A}^{-1}\\ &=e^{:h_4:}\hat{M}e^{:f_4:}e^{-:h_4:}\\ &=\hat{M}\hat{M}^{-1}e^{:h_4}\hat{M}e^{:f_4:}e^{-:h_4:}\\ &=\hat{M}e^{:\hat{M}^{-1}h_4:}e^{:f_4:}e^{-:h_4:}.\end{aligned}$$

Using the Campbell-Baker-Hausdorff series [8],

$$\mathcal{N}=\hat{M}\exp(:M^{-1}h_4-h_4+f_4:+\text{higher order terms}), \tag{4.17}$$

which gives (up to fourth order)

$$\mathcal{N}\sim\hat{M}\exp(:\hat{M}^{-1}h_4-h_4+f_4:). \tag{4.18}$$

The linear part $\hat{M}$ is already in the normal form. The next step is to simplify the Lie operator $:M^{-1}h_4-h_4+f_4:$, so that it is a function only of J_1. Let

$$h_4=a_{40}|4,0\rangle_R+a_{31}|3,1\rangle_R+a_{22}|2,2\rangle_R+b_{40}|4,0\rangle_I+b_{31}|3,1\rangle_I. \tag{4.19}$$

Also,

$$\hat{M}^{-1}|4,0\rangle_R=\cos 4t|4,0\rangle_R-\sin 4t|4,0\rangle_I,$$

$$\hat{M}^{-1}|3,1\rangle_R=\cos 2t|3,1\rangle_R-\sin 2t|3,1\rangle_I,$$

$$\hat{M}^{-1}|4,0\rangle_I=\cos 4t|4,0\rangle_I+\sin 4t|4,0\rangle_R,$$

$$\hat{M}^{-1}|3,1\rangle_I=\cos 2t|3,1\rangle_I+\sin 2t|3,1\rangle_R,$$

$$\hat{M}^{-1}|2,2\rangle_R=|2,2\rangle_R.$$

Therefore, we have

$$\begin{aligned}\hat{M}^{-1}h_4-h_4+f_4={}&(a_{40}(\cos 4t-1)+b_{40}\sin 4t+c_{40})|4,0\rangle_R+(-a_{40}\sin 4t+b_{40}(\cos 4t-1)+d_{40})|4,0\rangle_I+c_{22}|2,2\rangle_R\\&+(a_{31}(\cos 2t-1)+b_{31}\sin 2t+c_{31})|3,1\rangle_R+(-a_{31}\sin 2t+b_{31}(\cos 2t-1)+d_{31})|3,1\rangle_I.\end{aligned} \tag{4.20}$$

As expected, the basis element $|2,2\rangle_R$ cannot be removed. Therefore, a_{22} in Eq. (4.19) can be chosen as zero. The coefficients of the other basis elements in the above expression can be made zero by choosing a_{ij}'s and b_{ij}'s as follows:

$$\begin{aligned}&a_{40}(\cos 4t-1)+b_{40}\sin 4t=-c_{40},\\&-a_{40}\sin 4t+b_{40}(\cos 4t-1)=-d_{40},\\&a_{31}(\cos 2t-1)+b_{31}\sin 2t=-c_{31},\\&-a_{31}\sin 2t+b_{31}(\cos 2t-1)=-d_{31}.\end{aligned} \tag{4.21}$$

Thus the only nonzero term remaining in $\hat{M}^{-1}h_4-h_4+f_4$ is $c_{22}|2,2\rangle_R=c_{22}J_1^2$. So we have the normal form as

$$\mathcal{N}=\hat{M}e^{:c_{22}J_1^2:}. \tag{4.22}$$

Solving the above set of equations, we obtain

$$\begin{aligned}&a_{40}=\frac{1}{2(1-\cos 4t)}(-c_{40}(\cos 4t-1)+d_{40}\sin 4t),\\&b_{40}=\frac{1}{2(1-\cos 4t)}(-c_{40}\sin 4t-d_{40}(\cos 4t-1)),\\&a_{31}=-\frac{1}{2(1-\cos 2t)}(c_{31}(\cos 2t-1)-d_{31}\sin 2t),\\&b_{31}=-\frac{1}{2(1-\cos 2t)}(c_{31}\sin 2t+d_{31}(\cos 2t-1)),\\&a_{22}=0.\end{aligned} \tag{4.23}$$

Simplifying the above set of equations, we have

$$a_{40}=0,\ \ b_{40}=-\tfrac{1}{768},\ \ a_{31}=0,\ \ b_{31}=-\tfrac{1}{96}. \tag{4.24}$$

Thus we have succeeded in obtaining the transformation,

$$\mathcal{A}=e^{:h_4:}, \tag{4.25}$$

where h_4 is given by substituting the values of a_{ij}'s and b_{ij}'s in Eq. (4.19).

C. Dynamic moment invariants

We are now in a position to determine the dynamic moment invariants for the pendulum Hamiltonian. From Eq (3.7), the dynamic moment invariants of the pendulum maps are of the form

$$\mathcal{D}_k=e^{-:h_4:}\mathcal{I}_k, \tag{4.26}$$

where $\mathcal{I}_k$'s are the normal form moment invariants given in Sec. III.

As an example, we compute $\mathcal{D}_2^{(1)}$ for the pendulum Hamiltonian. From Eq. (4.19) (after converting to the usual monomial basis) we obtain

$$h_4=r_1q_1^4+r_2p_1^4+r_3q_1^3p_1+r_4q_1^2p_1^2+r_5q_1p_1^3, \tag{4.27}$$

where

$$\begin{aligned}&r_1=(a_{40}+a_{31})=0,\quad r_2=(a_{40}-a_{31})=0,\\&r_3=(4b_{40}+2b_{31})=-\tfrac{5}{192},\\&r_4=-6a_{40}=0,\quad r_5=-4b_{40}+2b_{31}=-\tfrac{1}{64}.\end{aligned} \tag{4.28}$$

Therefore, h_4 is given by

$$h_4=-\tfrac{5}{192}q_1^3p_1-\tfrac{1}{64}p_1^3q_1. \tag{4.29}$$

The functionally independent normal form moment invariant $\mathcal{I}_2^{(1)}$ is given by [cf. Eq. (3.32)]

$$\mathcal{I}_2^{(1)}=(\langle q_1^2\rangle+\langle p_1^2\rangle). \tag{4.30}$$

Therefore, up to fourth order, we have

$$
\begin{aligned}
e^{-:h_4:}\mathcal{I}_2^{(1)} &= e^{-:h_4:}(\langle q_1^2\rangle + \langle p_1^2\rangle) \\
&= (1 - :h_4: + \cdots)(\langle q_1^2\rangle + \langle p_1^2\rangle) \\
&= \langle q_1^2\rangle + \langle p_1^2\rangle - :h_4:\langle q_1^2\rangle - :h_4:\langle p_1^2\rangle + \cdots .
\end{aligned} \tag{4.31}
$$

Using Eq. (2.12), we obtain

$$:h_4:\langle q_1^2\rangle = \tfrac{5}{96}\langle q_1^4\rangle + \tfrac{3}{32}\langle p_1^2 q_1^2\rangle,$$

$$:h_4:\langle p_1^2\rangle = -\tfrac{5}{32}\langle q_1^2 p_1^2\rangle - \tfrac{1}{32}\langle p_1^4\rangle.$$

The dynamic moment invariant $\mathcal{D}_2^{(1)}$ is therefore given by

$$
\begin{aligned}
\mathcal{D}_2^{(1)} = e^{-:h_4:}\mathcal{I}_k = \langle q_1^2\rangle + \langle p_1^2\rangle - \tfrac{5}{96}\langle q_1^4\rangle + \tfrac{1}{32}\langle p_1^4\rangle + \tfrac{1}{16}\langle q_1^2 p_1^2\rangle \\
+ (\text{higher order terms}).
\end{aligned} \tag{4.32}
$$

Similarly, one can easily obtain other dynamic moment invariants for the pendulum Hamiltonian.

V. CONCLUSIONS

In this paper, we have outlined a procedure for constructing dynamic moment invariants for nonlinear Hamiltonian systems. This was made possible by the normal form techniques available for symplectic maps representing the Hamiltonian. We applied our method to the nonlinear pendulum Hamiltonian, and constructed dynamic moment invariants for this system, since the results can be explicitly stated in this case. This method can be readily applied to other nonlinear Hamiltonian systems once the corresponding symplectic map is known.

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